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Exact solution of an electron system combining two different t–J models

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Abstract. A new strongly correlated electron model is presented. This is formed by two types of sites: one where double occupancy is forbidden, as in the t–J model, and the other where double occupancy is allowed but vacancy is not allowed, as in an inverse t–J model. The Hamiltonian shows nearest and next-to-nearest neighbour interactions and it is solved by means of a modified algebraic nested Bethe ansatz. The number of sites where vacancy is not allowed, may be treated as a new parameter if the model is regarded as a t–J model with impurities. The ground and excited states are described in the thermodynamic limit.

1. Introduction

Strongly correlated systems are very interesting in view of their relation with high T_c superconductivity. It is very important to study the one-dimensional systems because they may share properties with two-dimensional ones [1]. The t–J model was proposed by Zhang and Rice [2] and it describes electrons on a lattice excluding the double occupancy of any site, in opposition to the Hubbard model [3], where the double occupancy is not forbidden. At the point $J = 2t$ the model (called supersymmetric) is associated to a graded Lie algebra and it is exactly integrable. Their integrability was studied in [4], although equivalent systems had been solved by other authors as in [5, 6]. The ground state and the excitation spectrum were investigated in [7, 8] and low-lying excitations close to half-filling were treated in [9]. The integration of the model by means of the nested algebraic Bethe ansatz (NABA) [10, 11] in the framework of the graded quantum inverse scattering method (GQISM) [12] was established in [13]. The completeness of the Bethe states was considered in [14] and the properties of the model in an external magnetic field were studied in [15].

In this paper we propose a model with two different kinds of sites. The sites of the first type may be unoccupied or occupied by an electron with spin up or down, but double occupancy is not allowed. We call these sites t–J. The sites of the second type may be occupied by an electron with spin up or down, or by two electrons with antiparallel spins in a singlet state, but vacancies are not possible. We call these sites 'J–t' because it is as an inverse t–J model. We regard these sites as frustrated Hubbard sites. Other t–J models with impurities have been proposed by different authors [16].

In order to make the inhomogeneous system we use the R-matrix of the t–J model and the graded Yang–Baxter equation (GYBE). We take another solution that we associate with

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the 'J–t' sites, which can give us another integrable homogeneous system. The system that we propose is formed by alternating N_h states in the t–J sites with N_p states in the J–t sites.

As we will see, the method is trivially dependent of the number of sites of each class, and then we diagonalize the transfer matrix for a general lattice with N_h and N_p states in the respectives sites, although we use an alternating chain with $N_h = N_p = N/2$ to compute the Hamiltonian. This Hamiltonian is the sum of nearest neighbour interaction terms (two-site operators) and next-to-nearest neighbour interaction terms (three-site operators). The diagonalization is made using the method that we proposed in [17]. This method is more general than the usual NABA, and it has been used by other authors in a different system [18]. Recently, Links and Foerster have proposed a model [19] with the same kind of alternating site states that in our model and they solve it with the same method.

As we said before, and will see as we proceed through this paper, all results that we have are independent of the number and position of the two kinds of sites, then our system can be considered as a t–J model with impurities J–t.

2. The model

The t–J model may be deduced, in the framework of the graded Lie algebras, by means of the L-operator. We are going to take the operators given by Eßler and Korepin in [13].

$$L^{(t-J)}(\lambda) = \lambda I + \mathrm{i}P \tag{2.1}$$

where

$$I^{a,b}_{c,d} = \delta_{a,b}\delta_{c,d} \tag{2.2a}$$

$$P^{a,b}_{c,d} = \delta_{a,d}\delta_{b,c}(-1)^{\epsilon_b\epsilon_d} \tag{2.2b}$$

and ϵ_j are the Grassmann parities of the basis vectors. We will use in this paper the Fermion–Fermion–Boson (FFB) basis, that is, $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = 0$. The upper indices are in the space that we call auxiliary, and the lower indices in the site space.

The operator $L^{(t-J)}$ verifies the GYBE

$$R(\lambda-\mu)[L^{(t-J)}(\lambda)\otimes L^{(t-J)}(\mu)] = [L^{(t-J)}(\mu)\otimes L^{(t-J)}(\lambda)]R(\lambda-\mu) \tag{2.3}$$

where the R-matrix is given by

$$R(\lambda) = \lambda P + \mathrm{i}I \tag{2.4}$$

and the tensor product is the graded tensor product, which is defined as

$$(F\otimes G)^{a,b}_{c,d} = F_{a,b}G_{c,d}(-1)^{\epsilon_c(\epsilon_a+\epsilon_b)}. \tag{2.5}$$

Equation (2.3) in components is,

$$\begin{aligned} R(\lambda-\mu)^{a,c_1}_{b,d_1} L^{(t-J)}(\lambda)^{c_1,c}_{g,e_1} L^{(t-J)}(\mu)^{d_1,d}_{e_1,h}(-1)^{\epsilon_{d_1}(\epsilon_{c_1}+\epsilon_c)} \\ = L^{(t-J)}(\mu)^{a,c_2}_{g,e_2} L^{(t-J)}(\lambda)^{b,d_2}_{e_2,h} R(\lambda-\mu)^{c_2,c}_{d_2,d}(-1)^{\epsilon_b(\epsilon_a+\epsilon_{c_2})}. \end{aligned} \tag{2.6}$$

In order to make a mixed lattice, we need another L-operator associated to the J–t sites. That operator must fulfil the same GYBE equation (2.3). Inspired by the same methods given in [13], we have found that

$$L^{(J-t)}(\lambda) = \left(\lambda - \frac{\mathrm{i}}{2}\right) I - \mathrm{i}Q \tag{2.7}$$

verifies our requirements, being

$$Q^{a,b}_{c,d} = \delta_{a,c}\delta_{b,d}(-1)^{\epsilon_a\epsilon_c}. \tag{2.8}$$

We have used the t–J basis

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{2.9}$$

and the J–t basis

$$|\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{2.10}$$

It is easy to show by direct calculation

$$L^{(J-t)}(\lambda)^{a,b}_{\alpha,\beta} L^{(J-t)}(-\lambda)^{\beta,\gamma}_{b,c} = \rho(\lambda)\delta_{a,c}\delta_{\alpha,\gamma} \tag{2.11}$$

with

$$\rho(\lambda) = -\tfrac{1}{4} - \lambda^2. \tag{2.12}$$

We build the monodromy matrix alternating the t–J and the J–t L-operators.

$$\begin{aligned} T_{a,b}(\lambda, w) = {} & L^{(t-J)}(\lambda)^{a,c_1}_{a_1,b_1} L^{(J-t)}(\lambda + w)^{c_1,c_2}_{\alpha_1,\beta_1} L^{(t-J)}(\lambda)^{c_2,c_3}_{a_2,b_2} \\ & \ldots L^{(t-J)}(\lambda)^{c_{N-2},c_{N-1}}_{a_{N/2},b_{N/2}} L^{(J-t)}(\lambda + w)^{c_{N-1},b}_{\alpha_{N/2},\beta_{N/2}} \\ & \times(-1)^{[\sum_{i=1}^{N/2}(a_i+b_i)\sum_{j=1}^{N/2}(\alpha_j+a_{j+1})+\sum_{i=1}^{N/2-1}(\alpha_i+\beta_i)\sum_{j=1}^{N/2-1}(a_{j+1}+\alpha_{j+1})]} \end{aligned} \tag{2.13}$$

with N even and $a_j = \alpha_j = 0$ if $j > N/2$. We have used the standard matrix product in the auxiliary space and the graded tensorial product in the site space.

In the auxiliary space we can write the monodromy matrix as

$$T(\lambda, w) = \begin{pmatrix} A_{1,1}(\lambda, w) & A_{1,2}(\lambda, w) & B_1(\lambda, w) \\ A_{2,1}(\lambda, w) & A_{2,2}(\lambda, w) & B_2(\lambda, w) \\ C_1(\lambda, w) & C_2(\lambda, w) & D(\lambda, w) \end{pmatrix}. \tag{2.14}$$

The transfer matrix is given by

$$F(\lambda, w) = \mathrm{strace}(T(\lambda, w)) = D(\lambda, w) - A_{1,1}(\lambda, w) - A_{2,2}(\lambda, w) \tag{2.15}$$

and the verification of (GYBE) by the monodromy matrix assures the commutation of the transfer matrices for different values of the argument, that is

$$[F(\lambda, w), F(\mu, w)] = 0. \tag{2.16}$$

The corresponding associate Hamiltonian is obtained by taking the first logarithmic derivative of the transfer matrix at equal zero spectral parameter

$$H(w) = -\mathrm{i}J \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln(F(\lambda, w))\Big|_{\lambda=0} \tag{2.17}$$

where J is a constant.

We write this Hamiltonian as the sum of both operators: the nearest neighbour interaction term and the next-to-nearest neighbour interaction term

$$H(w) = \frac{-\mathrm{i}J}{\rho(w)} \sum_{\substack{j=1 \\ j \text{ odd}}}^{N-1} h^{[1]}_{j,j+1} + \frac{-\mathrm{i}J}{c_1\rho(w)} \sum_{\substack{j=1 \\ j \text{ odd}}}^{N-1} h^{[2]}_{j,j+1,j+2} \tag{2.18}$$

with $R(0) = c_1 I$ and

$$(h^{[1]}_{j,j+1})_{a,b;\beta,\gamma} = \dot{L}^{(J-t)}(w)^{a,c}_{\beta,\delta} L^{(J-t)}(-w)^{\delta,\gamma}_{c,b} \tag{2.19a}$$

$$(h^{[2]}_{j,j+1,j+2})_{a,b;\beta,\gamma;c,d} = L^{(J-t)}(w)^{a,e}_{\beta,\delta} \dot{L}^{(t-J)}(0)^{e,d}_{c,f} L^{(J-t)}(-w)^{\delta,\gamma}_{f,b} (-1)^{\epsilon_c(\epsilon_\beta+\epsilon_\delta)}. \tag{2.19b}$$

The dot over the operator shows the derivative.

Taking $w = 0$ we find that,

$$
\begin{aligned}
H = J \sum_{\substack{j=1\\ j \text{ odd}}}^{N-1} \Bigg\{ & -2\Bigg\{ \sum_{\sigma=\uparrow,\downarrow} (c_{j,\sigma} d_{j+1,\sigma} - \text{h.c.}) - 2n_j \bar{n}_{j+1} \\
& + \sum_{\sigma=\uparrow,\downarrow} (d_{j+1,\sigma} c_{j+2,\sigma} - \text{h.c.}) - 2\bar{n}_{j+1} n_{j+2} \Bigg\} - 6n_j - 4\bar{n}_{j+1} \\
& +3\Bigg\{ \sum_{\sigma=\uparrow,\downarrow} (c^\dagger_{j,\sigma} c_{j+2,\sigma} + \text{h.c.}) + (S^-_j S^+_{j+2} + S^+_j S^-_{j+2} + 2S^z_j S^z_{j+2}) \Bigg\} - \tfrac{11}{2} n_j n_{j+2} \\
& -2\Bigg\{ \sum_{\sigma=\uparrow,\downarrow} \epsilon(\sigma)[(S^z_j d^\dagger_{j+1,\sigma} c^\dagger_{j+2,\sigma} - c^\dagger_{j,\sigma} S^z_{j+1} c_{j+2,\sigma} - c^\dagger_{j,\sigma} d^\dagger_{j+1,\sigma} S^z_{j+2}) + \text{h.c.}] \Bigg\} \\
& +3\Bigg\{ \Bigg[\sum_{\sigma=\uparrow,\downarrow} (c_{j,\sigma} d_{j+1,\sigma} - 3\text{h.c.}) \\
& -2(\tfrac{1}{4} n_j \bar{n}_{j+1} + S^-_j S^-_{j+1} + S^+_j S^+_{j+1} + 2S^z_j S^z_{j+1}) \Bigg] \cdot n_{j+2} \Bigg\} \\
& +n_j \cdot \Bigg[\sum_{\sigma=\uparrow,\downarrow} (d_{j+1,\sigma} c_{j+2,\sigma} - 3\text{h.c.}) \\
& +2(\tfrac{1}{4} \bar{n}_{j+1} n_{j+2} + S^-_{j+1} S^-_{j+2} + S^+_{j+1} S^+_{j+2} + 2S^z_{j+1} S^z_{j+2}) \Bigg] \\
& -2\Bigg\{ \sum_{\sigma=\uparrow,\downarrow} [(S^{\tau(\sigma)}_j d_{j+1,\sigma} c_{j+2,\bar{\sigma}} - c_{j,\sigma} S^{\tau(\sigma)}_{j+1} c_{j+2,\bar{\sigma}} - c^\dagger_{j,\sigma} d^\dagger_{j+1,\bar{\sigma}} S^{\tau(\sigma)}_{j+2}) + \text{h.c.}] \Bigg\} \\
& -\Bigg\{ 3 \sum_{\sigma=\uparrow,\downarrow} (c^\dagger_{j,\sigma} \bar{n}_{j+1} c_{j+2,\sigma} + \text{h.c.}) \\
& +2(S^-_j \bar{n}_{j+1} S^+_{j+2} + S^+_j \bar{n}_{j+1} S^-_{j+2} + 2S^z_j \bar{n}_{j+1} S^z_{j+2}) \Bigg\} \Bigg\} + 3N
\end{aligned}
\tag{2.20}
$$

where $c^\dagger$ and n are the creation operator and the number of electrons operator in the t–J sites, and $d^\dagger$ and $\bar{n}$ are the same operators in the J–t sites. Also, S is the spin operator. In terms of the elements of the L^{t-J} operators, they are

$$
\begin{aligned}
[c^\dagger_\uparrow]_{a,b} &= \frac{1}{\mathrm{i}} L^{(t-J)}(\lambda)^{3,1}_{a,b} \\
[c^\dagger_\downarrow]_{a,b} &= \frac{1}{\mathrm{i}} L^{(t-J)}(\lambda)^{3,2}_{a,b} \\
[c_\uparrow]_{a,b} &= \frac{1}{\mathrm{i}} L^{(t-J)}(\lambda)^{1,3}_{a,b} \\
[c_\downarrow]_{a,b} &= \frac{1}{\mathrm{i}} L^{(t-J)}(\lambda)^{2,3}_{a,b} \\
[S^+]_{a,b} &= -\frac{1}{\mathrm{i}} L^{(t-J)}(\lambda)^{2,1}_{a,b} \\
[S^-]_{a,b} &= -\frac{1}{\mathrm{i}} L^{(t-J)}(\lambda)^{1,2}_{a,b} \\
[S^z]_{a,b} &= \frac{\mathrm{i}}{2} \{L^{(t-J)}(\lambda)^{1,1}_{a,b} - L^{(t-J)}(\lambda)^{2,2}_{a,b}\}
\end{aligned}
\tag{2.21}
$$

$$[n]_{a,b} = -\mathrm{i}\frac{\lambda + \mathrm{i}}{\lambda - \mathrm{i}}\{L^{(t-J)}(\lambda)^{1,1}_{a,b} + L^{(t-J)}(\lambda)^{2,2}_{a,b}\} + \frac{2(\lambda^2 - \mathrm{i}\lambda + 1)}{\lambda(\mathrm{i}\lambda + 1)}L^{(t-J)}(\lambda)^{3,3}_{a,b}$$

and in terms of the L^{J-t}

$$\begin{aligned}
[d^\dagger_\uparrow]_{a,b} &= \frac{1}{\mathrm{i}}L^{(J-t)}(\lambda)^{1,3}_{a,b}\\
[d^\dagger_\downarrow]_{a,b} &= \frac{1}{\mathrm{i}}L^{(J-t)}(\lambda)^{2,3}_{a,b}\\
[d_\uparrow]_{a,b} &= -\frac{1}{\mathrm{i}}L^{(J-t)}(\lambda)^{3,1}_{a,b}\\
[d_\downarrow]_{a,b} &= -\frac{1}{\mathrm{i}}L^{(J-t)}(\lambda)^{3,2}_{a,b}\\
[S^+]_{a,b} &= \frac{1}{\mathrm{i}}L^{(J-t)}(\lambda)^{1,2}_{a,b}\\
[S^-]_{a,b} &= \frac{1}{\mathrm{i}}L^{(J-t)}(\lambda)^{2,1}_{a,b}\\
[S^z]_{a,b} &= -\frac{\mathrm{i}}{2}\{L^{(J-t)}(\lambda)^{1,1}_{a,b} - L^{(J-t)}(\lambda)^{2,2}_{a,b}\}\\
[\overline{n}]_{a,b} &= -\mathrm{i}\{L^{(J-t)}(\lambda)^{1,1}_{a,b} + L^{(J-t)}(\lambda)^{2,2}_{a,b}\} - \frac{2\lambda - 3\mathrm{i}}{2\lambda - 3\mathrm{i}}L^{(J-t)}(\lambda)^{3,3}_{a,b}.
\end{aligned} \tag{2.22}$$

We have also used the following notation:

$$\bar{\sigma} = \begin{cases} \downarrow & \text{if} \quad \sigma = \uparrow \\ \uparrow & \text{if} \quad \sigma = \downarrow \end{cases} \tag{2.23a}$$

$$\tau(\uparrow) = \downarrow \qquad \tau(\downarrow) = \uparrow \qquad \epsilon(\uparrow) = 1 \qquad \epsilon(\downarrow) = -1. \tag{2.23b}$$

Obviously, the Hamiltonian (2.20) is not Hermitian, however as we will see in section 4, it has real eigenvalues. [20, 21] show that for the supersymmetric t–J no Hermitian Hamiltonians with quantum group invariance enjoy this property. In a most general case, in [22] complex Hamiltonians with PT invariance is proof that they have real spectra.

3. Algebraic Bethe ansatz

The monodromy operator T verifies the GYBE, independently of the combination of t–J and J–t sites that we take in our system. Then, we are going to take N_h sites of the first type and N_p of the second. The space of states of the total system will be,

$$E = \bigotimes_i^{N_h} E_i^{t-J} \bigotimes_j^{N_p} E_j^{J-t} \tag{3.1}$$

where E_i^{t-J} is the space of states of the site i and E_j^{J-t} the space of states of the site j.

In order to diagonalize the Hamiltonian by means of the nested algebraic Bethe ansatz (NABA), we should need to find a state of the system that verifies [12],

$$A_{i,j}\|v\rangle \propto \delta_{i,j}\|v\rangle \tag{3.2}$$

but this is not the case. To overcome the problem, we must follow a modified NABA (MNABA), which is described in [17]. For this purpose, in a first step, we build the vacuum subspace

$$\Omega = \bigotimes_i^{N_h} |0\rangle_i \bigotimes_j^{N_p} \{e\}_j \tag{3.3}$$

where $\{e\}_j$ denotes the subspace generated by the vectors $|\downarrow\rangle$ and $|\uparrow\rangle$ in a J–t site j. Any $\|w\rangle$ vector in the vacuum subspace verifies

$$A_{i,j}(\lambda)\|w\rangle \in \Omega \qquad i, j = 1, 2 \tag{3.4a}$$

$$C_i(\lambda)\|w\rangle \neq 0 \qquad i = 1, 2 \tag{3.4b}$$

$$B_i(\lambda)\|w\rangle = 0 \qquad i = 1, 2 \tag{3.4c}$$

$$D(\lambda)\|w\rangle = [a'(\lambda)]^{N_0}[b_1(\lambda)]^{N_p}\|w\rangle \tag{3.4d}$$

where $\|w\rangle \in \Omega$ and

$$a'(\lambda) = \lambda + \mathrm{i} \qquad b_1(\lambda) = \lambda - \tfrac{1}{2}. \tag{3.5}$$

From the GYBE we get the relation of commutation

$$A_{a,b}(\mu)C_c(\lambda) = (-1)^{\epsilon_a\epsilon_p} g(\mu-\lambda)r(\mu-\lambda)^{d,c}_{p,b}C_p(\lambda)A_{a,d}(\mu) + h(\mu-\lambda)C_b(\mu)A_{a,c}(\lambda) \tag{3.6a}$$

$$D(\mu)C_c(\lambda) = g(\lambda-\mu)C_c(\lambda)D(\mu) - h(\lambda-\mu)C_c(\mu)D(\lambda) \tag{3.6b}$$

$$C_a(\lambda)C_b(\mu) = r(\lambda-\mu)^{d,b}_{c,a}C_c(\mu)C_d(\lambda) \tag{3.6c}$$

where

$$g(\mu) = \frac{\mu + \mathrm{i}}{\mu} \tag{3.7a}$$

$$h(\mu) = \frac{\mathrm{i}}{\mu} \tag{3.7b}$$

$$r(\mu)^{a,b}_{c,d} = \frac{h(\mu)}{g(\mu)}\delta_{a,b}\delta_{c,d} - \frac{1}{g(\mu)}\delta_{a,d}\delta_{b,c}. \tag{3.7c}$$

In order to solve the equation

$$F(\lambda)\Psi = \Lambda(\lambda)\Psi \tag{3.8}$$

we build the state,

$$\Psi(\vec{\lambda}) \equiv \Psi(\lambda_1, \ldots, \lambda_r) = C_{a_1}(\lambda_1)\ldots C_{a_r}(\lambda_r)X^{a_1\ldots a_r}\|1\rangle \tag{3.9}$$

with $\|1\rangle \in \Omega$. When we apply the $D(\mu)$ operator to Ψ, using (3.6a)–(3.6c), we push this operator to the right of the C operators, and we get a wanted term characterized by the C operators conserve their arguments, and several unwanted terms characterized by the arguments of the C operators are interchanged. The wanted term is,

$$[a'(\mu)]^{N_0}[b_1(\mu)]^{N_p}\prod_{j=1}^{r} g(\lambda_j - \mu)\Psi \tag{3.10}$$

and the kth unwanted term is,

$$\begin{aligned}-h(\lambda_k-\mu)[a'(\lambda_k)]^{N_0}[b_1(\lambda_k)]^{N_p}\prod_{j=1}^{r} g(\lambda_j-\lambda_k)C_{b_k}(\mu)C_{b_{k+1}}(\lambda_{k+1})\\ \ldots C_{b_{k-1}}(\lambda_{k-1})M^{(k-1)}(\lambda_{k-1})^{b_1,\ldots,b_r}_{a_1,\ldots,a_r}X^{a_1,\ldots,a_r}\|1\rangle\end{aligned} \tag{3.11}$$

with M given in appendix A.

The application of $A_{a,a}$ to the state Ψ is a little larger but straightforward. We again get wanted and unwanted terms, and after some calculations we find the wanted term,

$$\prod_{j=1}^{r} g(\mu-\lambda_j)C_{p_1}(\lambda_1)\ldots C_{p_r}(\lambda_r)A_{a,b}(\mu)[Z(\mu,\vec{\lambda})^{p_1,\ldots,p_r}_{a_1,\ldots,a_r}]_{b,a}X^{a_1,\ldots,a_r}\|1\rangle \tag{3.12}$$

where the Z operator is

$$[Z(\mu,\vec{\lambda})^{p_1,\ldots,p_r}_{a_1,\ldots,a_r}]_{i,j} = L^{(1)}(\mu-\lambda_r)^{i,d_{r-1}}_{p_r,a_r} L^{(1)}(\mu-\lambda_{r-1})^{d_{r-1},d_{r-2}}_{p_{r-1},a_{r-1}} \ldots L^{(1)}(\mu-\lambda_2)^{d_2,d_1}_{p_2,a_2} L^{(1)}(\mu-\lambda_1)^{d_1,j}_{p_1,a_1} \tag{3.13}$$

with

$$L^{(1)}(\lambda)^{f,d}_{e,b} = r(\lambda)^{a,b}_{c,d} P^{(1)}{}^{e,a}_{f,c} \tag{3.14}$$

and $P^{(1)}{}^{e,a}_{f,c} = \delta_{e,c}\delta_{a,f}(-1)^{\epsilon_a\epsilon_c}$.

Due to $j=1$ or $j=2$, we have $\epsilon_j = 1$, then (3.14) becomes

$$L^{(1)}(\lambda)^{f,d}_{e,b} = -r(\lambda)^{f,b}_{e,d}. \tag{3.15}$$

Now we will get ready to prepare the second step of the MNABA. We define the monodromy matrix at second level as

$$T^{(2)}(\mu,\vec{\lambda}) = A(\mu)\cdot Z(\mu,\vec{\lambda}). \tag{3.16}$$

In the auxiliary space, that is now two-dimensional, this operator can be written as

$$T^{(2)}(\mu,\vec{\lambda}) = \begin{pmatrix} A^{(2)}(\mu,\vec{\lambda}) & C^{(2)}(\mu,\vec{\lambda}) \\ B^{(2)}(\mu,\vec{\lambda}) & D^{(2)}(\mu,\vec{\lambda}) \end{pmatrix}. \tag{3.17}$$

The transfer matrix at second level is

$$F^{(2)}(\mu,\vec{\lambda}) = \mathrm{strace}[T^{(2)}(\mu,\vec{\lambda})] = -A^{(2)}(\mu,\vec{\lambda}) - D^{(2)}(\mu,\vec{\lambda}). \tag{3.18}$$

Now, the wanted term (3.12) can be written as,

$$-\prod_{j=1}^{r} g(\mu-\lambda_j) C_{p_1}(\lambda_1)\ldots C_{p_r}(\lambda_r) F^{(2)}(\mu,\vec{\lambda})^{p_1,\ldots,p_r}_{a_1,\ldots,a_r} X^{a_1,\ldots,a_r}\|1\rangle \tag{3.19}$$

and the kth unwanted term as

$$h(\mu-\lambda_k)\prod_{\substack{j=1\\ j\neq k}}^{r} g(\lambda_k-\lambda_j) C_{p_k}(\mu) C_{p_{k+1}}(\lambda_{k+1})\ldots C_{p_r}(\lambda_r) C_{p_1}(\lambda_1) \ldots C_{p_{k-1}}(\lambda_{k-1}) M^{(k-1)}(\lambda_{k-1})^{p_1,\ldots,p_r}_{b_1,\ldots,b_r} F^{(2)}(\lambda_k,\vec{\lambda})^{b_1,\ldots,b_r}_{a_1,\ldots,a_r} X^{a_1,\ldots,a_r}\|1\rangle. \tag{3.20}$$

We have found a new problem; we have to solve the equation,

$$F^{(2)}(\mu,\vec{\lambda}) X\|1\rangle = \Lambda^{(2)}(\mu,\vec{\lambda}) X\|1\rangle. \tag{3.21}$$

For this purpose we use the fact that $L^{(1)}(\lambda)$ verifies the GYBE with the r-matrix, and then we can solve (3.21) following the same path applied before.

The new vacuum is

$$\|1'\rangle = \bigotimes_{i}^{N} |0\downarrow\rangle_i \bigotimes_{j}^{r} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.22}$$

where the state $|0\downarrow\rangle$ is composed by a state $|0\rangle$ in the t–J sites and $|\downarrow\rangle$ in the J–t sites. The first tensorial product is the space where $A(\mu)$ works, whereas $Z(\mu,\vec{\lambda})$ works in the second one. This vacuum verifies

$$B^{(2)}(\mu,\vec{\lambda})\|1'\rangle = 0 \tag{3.23a}$$

$$C^{(2)}(\mu,\vec{\lambda})\|1'\rangle \neq 0 \tag{3.23b}$$

$$A^{(2)}(\mu,\vec{\lambda})\|1'\rangle = [b(\mu)]^{N_0}[b_1(\mu)]^{N_p}\prod_{j=1}^{r}\frac{g(\lambda_j-\mu)}{g(\mu-\lambda_j)}\|1'\rangle \tag{3.23c}$$

$$D^{(2)}(\mu,\vec{\lambda})\|1'\rangle = [b(\mu)]^{N_0}[a_1(\mu)]^{N_p}\prod_{j=1}^{r}\frac{1}{g(\mu-\lambda_j)}\|1'\rangle \tag{3.23d}$$

with

$$a_1(\lambda) = \lambda + \frac{\mathrm{i}}{2} \qquad b(\lambda) = \lambda. \tag{3.24}$$

The new relations of commutation are obtained from the GYBE for $T^{(2)}$

$$D^{(2)}(\mu)C^{(2)}(\lambda) = g(\lambda - \mu)C^{(2)}(\lambda)D^{(2)}(\mu) + h(\mu - \lambda)C^{(2)}(\mu)D^{(2)}(\lambda) \tag{3.25a}$$

$$A^{(2)}(\mu)C^{(2)}(\lambda) = g(\mu - \lambda)C^{(2)}(\lambda)A^{(2)}(\mu) + h(\lambda - \mu)C^{(2)}(\mu)A^{(2)}(\lambda). \tag{3.25b}$$

In the second level, we build the state

$$X\|1\rangle \equiv \Psi^{(2)}(\mu_1, \ldots, \mu_s) = C^{(2)}(\mu_1)\ldots C^{(2)}(\mu_s)\|1'\rangle. \tag{3.26}$$

Applying $A^{(2)}$ to this state we get a wanted term

$$[b(\mu)]^{N_0}[b_1(\mu)]^{N_p} \prod_{i=1}^{s} g(\mu - \mu_i) \prod_{j=1}^{r} \frac{g(\lambda_j - \mu)}{g(\mu - \lambda_j)} \Psi^{(2)} \tag{3.27}$$

and unwanted terms of which, the kth is

$$\begin{aligned} &h(\mu_k - \mu)[b(\mu_k)]^{N_0}[b_1(\mu_k)]^{N_p} \prod_{\substack{i=1\\ i\neq k}}^{s} g(\mu_k - \mu_i) \\ &\qquad \times \prod_{j=1}^{r} \frac{g(\lambda_j - \mu_k)}{g(\mu_k - \lambda_j)} C^{(2)}(\mu)C^{(2)}(\mu_{k+1})\ldots C^{(2)}(\mu_{k-1})\|1'\rangle. \end{aligned} \tag{3.28}$$

Also for $D^{(2)}$ we get a wanted and unwanted, the wanted is,

$$[b(\mu)]^{N_0}[a_1(\mu)]^{N_p} \prod_{i=1}^{s} g(\mu_i - \mu) \prod_{j=1}^{r} \frac{1}{g(\mu - \lambda_j)} \Psi^{(2)} \tag{3.29}$$

and the kth unwanted term is,

$$\begin{aligned} &h(\mu - \mu_k)[b(\mu_k)]^{N_0}[a_1(\mu_k)]^{N_p} \prod_{\substack{i=1\\ i\neq k}}^{s} g(\mu_i - \mu_k) \\ &\qquad \times \prod_{j=1}^{r} \frac{1}{g(\mu_k - \lambda_j)} C^{(2)}(\mu)C^{(2)}(\mu_{k+1})\ldots C^{(2)}(\mu_{k-1}). \end{aligned} \tag{3.30}$$

The cancellation of the unwanted terms, (3.11) with (3.20) and (3.28) with (3.30), gives us the ansatz equations

$$\left[\frac{b(\lambda_k)}{a'(\lambda_k)}\right]^{N_0} = \prod_{i=1}^{s} \frac{1}{g(\lambda_k - \mu_i)} \qquad k = 1, \ldots, r \tag{3.31a}$$

$$\left[\frac{a_1(\mu_n)}{b_1(\mu_n)}\right]^{N_p} = \prod_{j=1}^{r} g(\lambda_j - \mu_n) \prod_{\substack{i=1\\ i\neq n}}^{s} \frac{g(\mu_n - \mu_i)}{g(\mu_i - \mu_n)} \qquad n = 1, \ldots, s. \tag{3.31b}$$

On the other hand, collecting the wanted terms we get the eigenvalue of the transfer matrix,

$$\begin{aligned} \Lambda(\mu) = {} & [a'(\mu)]^{N_0}[b_1(\mu)]^{N_p} \prod_{j=1}^{r} g(\lambda_j - \mu) - [b(\mu)]^{N_0} \\ & \times \left\{ [b_1(\mu)]^{N_p} \prod_{i=1}^{s} g(\mu - \mu_i) \prod_{j=1}^{r} g(\lambda_j - \mu) + [a_1(\mu)]^{N_p} \prod_{i=1}^{s} g(\mu_i - \mu) \right\}. \end{aligned} \tag{3.32}$$

4. Thermodynamics of the model

The eigenstates of the transfer matrix can be characterized by observables operators that commute with it, and then with the Hamiltonian. We have found that the following observables

$$O_1 = \sum_{\substack{i=1 \\ i \text{ odd}}}^{N-1} (n_i + \bar{n}_{i+1}) \tag{4.1a}$$

$$O_2 = \sum_{\substack{i=1 \\ i \text{ odd}}}^{N-1} (n_i^{\text{holes}} - \bar{n}_{i+1}^{\text{pairs}}) \tag{4.1b}$$

$$O_3 = \sum_{\substack{i=1 \\ i \text{ odd}}}^{N-1} (n_{i,\uparrow} + \bar{n}_{i+1,\downarrow}) \tag{4.1c}$$

$$O_4 = \sum_{\substack{i=1 \\ i \text{ odd}}}^{N-1} (n_{i,\downarrow} + \bar{n}_{i+1,\uparrow}) \tag{4.1d}$$

commute with the transfer matrix

$$[O_i, F(\lambda)] = 0 \qquad i = 1, \ldots, 4. \tag{4.2}$$

We must note that O_1 is the total number of the electrons, O_2 is the difference between the number of holes and number of electron pairs, O_3 is the number of electrons with the spin up in the $\{|0\rangle, |\downarrow\rangle, |\uparrow\rangle\}$ base plus the number of electrons with the spin down in the $\{|\downarrow, \uparrow\rangle, |\downarrow\rangle, |\uparrow\rangle\}$ base and O_4 is the complimeny of O_3. Only two of these quantities are independent, the others are linearly dependent and thus, we are going to use O_1 and O_3.

The commutation relations of the C_i creation operators with these observables allows us to relate their eigenvalues with the numbers r and s of the ansatz equations. They are:

$$[O_1, C_i] = C_i \tag{4.3a}$$

$$[O_2, C_i] = -C_i \tag{4.3b}$$

$$[O_3, C_i] = \delta_{1,i} C_i \tag{4.3c}$$

$$[O_4, C_i] = \delta_{2,i} C_i \tag{4.3d}$$

and a long but straighforward calculation, gives us the action of these operators on the eigenstate $\psi(\lambda_1, \ldots, \lambda_r)$,

$$O_1 \psi(\lambda_1, \ldots, \lambda_r) = (r + N_p)\psi(\lambda_1, \ldots, \lambda_r) \tag{4.4a}$$

$$O_2 \psi(\lambda_1, \ldots, \lambda_r) = (-r + N_h)\psi(\lambda_1, \ldots, \lambda_r) \tag{4.4b}$$

$$O_3 \psi(\lambda_1, \ldots, \lambda_r) = (r - s + N_p)\psi(\lambda_1, \ldots, \lambda_r) \tag{4.4c}$$

$$O_4 \psi(\lambda_1, \ldots, \lambda_r) = s\psi(\lambda_1, \ldots, \lambda_r). \tag{4.4d}$$

The next step is to solve the equations of the ansatz in the thermodynamic limit, i.e., when N, N_h and N_p are infinity, but N_h/N and N_p/N remain finite. For this, it is convenient to reparametrize the roots as follows:

$$\lambda_j = v_j^{(1)} - \frac{\mathrm{i}}{2} \qquad j = 1, \ldots, r \tag{4.5a}$$

$$\mu_k = v_k^{(2)} \qquad k = 1, \ldots, s \tag{4.5b}$$

and then the equations (3.31*a*), (3.31*b*) are written

$$\left[\frac{v_k^{(1)}-\frac{\mathrm{i}}{2}}{v_k^{(1)}+\frac{\mathrm{i}}{2}}\right]^{N_h}=\prod_{j=1}^{s}\frac{v_k^{(1)}-v_j^{(2)}-\frac{\mathrm{i}}{2}}{v_k^{(1)}-v_j^{(2)}+\frac{\mathrm{i}}{2}} \tag{4.6a}$$

$$\left[\frac{v_k^{(2)}+\frac{\mathrm{i}}{2}}{v_k^{(2)}-\frac{\mathrm{i}}{2}}\right]^{N_p}=-\prod_{j=1}^{r}\frac{v_k^{(2)}-v_j^{(1)}-\frac{\mathrm{i}}{2}}{v_k^{(2)}-v_j^{(1)}+\frac{\mathrm{i}}{2}}\prod_{l=1}^{s}\frac{v_k^{(2)}-v_l^{(2)}+\mathrm{i}}{v_k^{(2)}-v_l^{(2)}-\mathrm{i}}. \tag{4.6b}$$

The energy is obtained with

$$E=-\mathrm{i}J\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\Lambda(\lambda)\bigg|_{\lambda=0}=-\mathrm{i}J\left[N_h\frac{\dot{a}'(0)}{a'(0)}+N_p\frac{\dot{b}_1(0)}{b_1(0)}-\sum_{j=1}^{r}\frac{\dot{g}(\lambda_j)}{g(\lambda_j)}\right] \tag{4.7}$$

and using (4.5)

$$E=J\left[-N_h+2N_p+\sum_{j=1}^{r}\frac{1}{(v_j^{(1)})^2+\frac{1}{4}}\right]. \tag{4.8}$$

It is convenient to introduce the function,

$$\phi(x)\equiv 2\arctan(x)=-\mathrm{i}\ln\frac{1+\mathrm{i}x}{1-\mathrm{i}x} \tag{4.9}$$

and taking logarithms in the equations of the ansatz (4.6*a*), (4.6*b*), we can write

$$N_h\phi(2v_k^{(1)})-\sum_{j}^{s}\phi(2v_k^{(1)}-2v_j^{(2)})=2\pi I_k^{(1)} \tag{4.10a}$$

$$N_p\phi(2v_k^{(2)})+\sum_{j}^{r}\phi(2v_k^{(2)}-2v_j^{(1)})-\sum_{l}^{s}\phi(v_k^{(2)}-v_l^{(1)})=2\pi I_k^{(2)} \tag{4.10b}$$

with $I_k^{(1)}$ and $I_k^{(2)}$ integers or half-odd integers and each set of these numbers determines a solution for roots $v_k^{(1)}$ and $v_k^{(2)}$.

In solving a BAE set, we have real roots and complex conjugate roots grouped in clusters that we call n-strings. The solution for the ground state in the thermodynamic limit can be obtained minimizing the free energy distribution [23]. In the Heisenberg model with arbitrary spin it is found that the solution for the ground state is formed by two-string roots [24]. In the t–J model, the two-level roots are mixed, they proliferate rapidly and it becomes difficult to determinate the roots which parametrize the ground state. Numerical analysis suggests that the structure of solutions in the ground state are two-string roots [8]. We assume the same hypothesis in our model.

Then, the solution in the ground state is given by two-string roots and for the lower excited states some roots are real. The equations (4.10*a*), (4.10*b*) can be reparametrized supposing that, in the r roots of the first level, $v_j^{(1)}$, $j=1,\ldots,r$, there are $2r_1$ roots of the form

$$v_j^{(1)}=\epsilon_l\pm\frac{\mathrm{i}}{2}\qquad l=1,\ldots,r_1 \tag{4.11}$$

with ϵ_l real, that are grouped in r_1 two-strings. The rest $r-2r_1$ roots are taken to be real and they are designed by,

$$v_j^{(1)}=\eta_l\qquad l=1,\ldots,(r-2r_1). \tag{4.12}$$

With the same assumptions, the s roots $v_j^{(2)}$ of the second level are written,

$$\begin{aligned}v_j^{(2)}&=\beta_l\pm\frac{\mathrm{i}}{2}\qquad l=1,\ldots,s_1\\ v_j^{(2)}&=\nu_l\qquad l=1,\ldots,(s-2s_1)\end{aligned} \tag{4.13}$$

with β_l and ν_l both real.

Taking this into account, the equations (4.10*a*), (4.10*b*) result for the respective roots.

$$N_h\phi(2\eta_k) - \sum_{i=1}^{s-2s_1}\phi(2\eta_k - 2\nu_i) - \sum_{i=1}^{s_1}\phi(\eta_k - \beta_i) = 2\pi I_k^{(1)} \tag{4.14a}$$

$$N_h\phi(\epsilon_k) - \sum_{i=1}^{s-2s_1}\phi(\epsilon_k - \nu_i) - \sum_{i=1}^{s_1}[\phi(2\epsilon_k - 2\beta_i) + \phi(\tfrac{2}{3}\epsilon_k - \tfrac{2}{3}\beta_i)] = 2\pi J_k^{(1)} \tag{4.14b}$$

$$N_p\phi(2\nu_k) + \sum_{j=1}^{r-2r_1}\phi(2\nu_k - 2\eta_j) + \sum_{j=1}^{r_1}\phi(\nu_k - \epsilon_j) - \sum_{i=1}^{s-2s_1}\phi(\nu_k - \nu_i)$$
$$- \sum_{i=1}^{s_1}[\phi(2\nu_k - 2\beta_i) + \phi(\tfrac{2}{3}\nu_k - \tfrac{2}{3}\beta_i)] = 2\pi I_k^{(2)} \tag{4.14c}$$

$$N_p\phi(\beta_k) + \sum_{j=1}^{r-2r_1}\phi(\beta_k - \eta_j) + \sum_{j=1}^{r_1}[\phi(2\beta_k - 2\epsilon_j) + \phi(\tfrac{2}{3}\beta_k - \tfrac{2}{3}\epsilon_j)]$$
$$- \sum_{i=1}^{s-2s_1}[\phi(2\beta_k - 2\nu_i) + \phi(\tfrac{2}{3}\beta_k - \tfrac{2}{3}\nu_i)] - \sum_{i=1}^{s_1}[2\phi(\beta_k - \beta_i) + \phi(\tfrac{1}{2}(\beta_k - \beta_i))]$$
$$= 2\pi J_k^{(2)} \tag{4.14d}$$

where we have used the identities in appendix B.

In the thermodynamic limit the system is described in the language of distribution functions of roots in both levels with particles and holes. In our case we define it as follows:

$$\rho_1 = \lim_{N\to\infty}\frac{1}{N(\eta_{j+1} - \eta_j)} \qquad \sigma_1 = \lim_{N\to\infty}\frac{1}{N(\epsilon_{j+1} - \epsilon_j)}$$
$$\rho_2 = \lim_{N\to\infty}\frac{1}{N(\rho_{j+1} - \rho_j)} \qquad \sigma_2 = \lim_{N\to\infty}\frac{1}{N(\beta_{j+1} - \beta_j)} \tag{4.15}$$

and we call $[B_1]$, $[B_2]$, $[C_1]$ and $[C_2]$ the regions where ρ_1 and ρ_2, σ_1 and σ_2 are defined respectively.

As usual [25], we introduce the functions,

$$Z_{\rho_1}(\lambda) = \frac{1}{2\pi N}\left[N_h\phi(2\lambda) - \sum_{i=1}^{s-2s_1}\phi(2\lambda - 2\nu_i) - \sum_{i=1}^{s_1}\phi(\lambda - \beta_i)\right] \tag{4.16a}$$

$$Z_{\sigma_1}(\lambda) = \frac{1}{2\pi N}\left[N_h\phi(\lambda) - \sum_{i=1}^{s-2s_1}\phi(\lambda - \nu_i) - \sum_{i=1}^{s_1}[\phi(2\lambda - 2\beta_i) + \phi(\tfrac{2}{3}\lambda - \tfrac{2}{3}\beta_i)]\right] \tag{4.16b}$$

$$Z_{\rho_2}(\lambda) = \frac{1}{2\pi N}\left[N_p\phi(2\lambda) + \sum_{j=1}^{r-2r_1}\phi(2\lambda - 2\eta_j) + \sum_{j=1}^{r_1}\phi(\lambda - \epsilon_j) - \sum_{i=1}^{s-2s_1}\phi(\lambda - \nu_i)\right.$$
$$\left. - \sum_{i=1}^{s_1}[\phi(2\lambda - 2\beta_i) + \phi(\tfrac{2}{3}\lambda - \tfrac{2}{3}\beta_i)]\right] \tag{4.16c}$$

$$Z_{\sigma_2}(\lambda) = \frac{1}{2\pi N}\left[N_p\phi(\lambda) + \sum_{j=1}^{r-2r_1}\phi(\lambda - \eta_j) + \sum_{j=1}^{r_1}[\phi(2\lambda - 2\epsilon_j) + \phi(\tfrac{2}{3}\lambda - \tfrac{2}{3}\epsilon_j)]\right.$$
$$\left. - \sum_{i=1}^{s-2s_1}[\phi(2\lambda - 2\nu_i) + \phi(\tfrac{2}{3}\lambda - \tfrac{2}{3}\nu_i)] - \sum_{i=1}^{s_1}[2\phi(\lambda - \beta_i) + \phi(\tfrac{1}{2}(\lambda - \beta_i))]\right]. \tag{4.16d}$$

These functions are monotonically increasing in their respective definition regions $[B_{1/2}]$ and $[C_{1/2}]$, and their values are integers when λ takes the value of a root. Besides, there are other values of λ where the Z functions take a integer or a half-odd integer value, but do not correspond to a root. We call each of these values a hole. The distribution functions are the derivatives of the Z functions [25],

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} Z_i(\lambda) \approx \frac{N}{N_{h/p}} \rho_i(\lambda) + \frac{1}{N_{h/p}} \sum_{i=1} \delta(\lambda - \theta_i) \tag{4.17}$$

where θ_i are the position of the corresponding holes. The thermodynamic limit is obtained by doing,

$$\lim_{N\to\infty} \frac{1}{N} \sum_i f(\lambda_i) \approx \int \mathrm{d}\lambda\, f(\lambda)\rho(\lambda). \tag{4.18}$$

Then, from (4.16*a*)–(4.16*d*), the corresponding distribution functions in this limit are,

$$\frac{N}{N_h}\rho_1(\lambda) = \frac{1}{2\pi}\left[2\phi'(2\lambda) - \frac{2N}{N_h}\int_{[B_2]} \phi'(2\lambda - 2\mu)\rho_2(\mu)\,\mathrm{d}\mu - \frac{N}{N_h}\int_{[C_2]} \phi'(\lambda-\mu)\sigma_2(\mu)\,\mathrm{d}\mu\right] \tag{4.19a}$$

$$\begin{aligned}\frac{N}{N_h}\sigma_1(\lambda) = \frac{1}{2\pi}\bigg[&2\phi'(\lambda) - \frac{N}{N_h}\int_{[B_2]} \phi'(\lambda-\mu)\rho_2(\mu)\,\mathrm{d}\mu \\ &-\frac{2N}{N_h}\int_{[C_2]} [\phi'(2(\lambda-\mu)) + \tfrac{1}{3}\phi'(\tfrac{2}{3}(\lambda-\mu))]\sigma_2(\mu)\,\mathrm{d}\mu\bigg]\end{aligned} \tag{4.19b}$$

$$\begin{aligned}\frac{N}{N_p}\rho_2(\lambda) = \frac{1}{2\pi}\bigg[&2\phi'(2\lambda) + \frac{2N}{N_p}\int_{[B_1]} \phi'(2\lambda-2\mu)\rho_1(\mu)\,\mathrm{d}\mu \\ &+\frac{N}{N_p}\int_{[C_1]} \phi'(\lambda-\mu)\sigma_1(\mu)\,\mathrm{d}\mu - \frac{N}{N_p}\int_{[B_2]} \phi'(\lambda-\mu)\rho_2(\mu)\,\mathrm{d}\mu \\ &-\frac{2N}{N_p}\int_{[C_2]} [\phi'(2(\lambda-\mu)) + \tfrac{1}{3}\phi'(\tfrac{2}{3}(\lambda-\mu))]\sigma_2(\mu)\,\mathrm{d}\mu\bigg]\end{aligned} \tag{4.19c}$$

$$\begin{aligned}\frac{N}{N_h}\sigma_2(\lambda) = \frac{1}{2\pi}\bigg[&\phi'(\lambda) + \frac{N}{N_p}\int_{[B_1]} \phi'(\lambda-\mu)\rho_1(\mu)\,\mathrm{d}\mu \\ &+\frac{2N}{N_p}\int_{[C_1]} [\phi'(2(\lambda-\mu)) + \tfrac{1}{3}\phi'(\tfrac{2}{3}(\lambda-\mu))]\sigma_1(\mu)\,\mathrm{d}\mu \\ &+\frac{2N}{N_p}\int_{[B_2]} [\phi'(2(\lambda-\mu)) + \tfrac{1}{3}\phi'(\tfrac{2}{3}(\lambda-\mu))]\rho_2(\mu)\,\mathrm{d}\mu \\ &-\frac{N}{N_p}\int_{[C_1]} [2\phi'(\lambda-\mu) + \tfrac{1}{2}\phi'(\tfrac{1}{2}(\lambda-\mu))]\sigma_2(\mu)\,\mathrm{d}\mu\bigg].\end{aligned} \tag{4.19d}$$

Quantities corresponding to observables defined before, can be found on this state in the thermodynamic limit in the same form, so the energy is given by

$$\frac{E}{N} = J\left[\frac{-N_h + 2N_p}{N} + 2\int_{[B_1]} \phi'(2\mu)\rho_1(\mu)\,\mathrm{d}\mu + \int_{[C_1]} \phi'(\mu)\sigma_1(\mu)\,\mathrm{d}\mu\right]. \tag{4.20}$$

The O_1 index (number of electrons) defined in (4.1*a*) is given by

$$n_e \equiv= \frac{O_1}{N} = \frac{r + N_p}{N} = \frac{N_p}{N} + \int_{[B_1]} \rho_1(\mu)\,\mathrm{d}\mu + \int_{[C_1]} \sigma_1(\mu)\,\mathrm{d}\mu. \tag{4.21}$$

The difference of magnetization S^z between the t–J sites minus the magnetization in the J–t sites (4.1c), (4.1d),

$$s^z = \frac{S^z_{t-J} - S^z_{J-t}}{N} = \frac{O_3 - O_4}{2N} = \frac{1}{2N}(r + N_p - 2s)$$
$$= \frac{n_e}{2} - \int_{[B_2]} \rho_2(\mu)\,\mathrm{d}\mu - 2\int_{[C_2]} \sigma_2(\mu)\,\mathrm{d}\mu. \qquad (4.22)$$

Using (4.19b), (4.19c) and (4.20), we find an important relation,

$$\frac{E}{N} = J\left[\frac{N_h - 2N_p}{N} - 2\pi(\sigma_1(0) - \rho_2(0))\right]. \qquad (4.23)$$

In our model, as we said before, we are going to suppose that the configuration of the ground state is given by two-string roots in both levels of the BAE, as it is suggested for the t–J model in [8]. With this hypothesis, since $\rho_1 = \rho_2 = 0$, the equations (4.19a)–(4.19d) are reduced to two equations and are simplified considerably,

$$\frac{N}{N_h}\sigma_1(\lambda) = \frac{1}{2\pi}\left[2\phi'(\lambda) - \frac{2N}{N_h}\int_{[C_2]}[\phi'(2(\lambda-\mu)) + \tfrac{1}{3}\phi'(\tfrac{2}{3}(\lambda-\mu))]\sigma_2(\mu)\,\mathrm{d}\mu\right] \qquad (4.24a)$$

$$\frac{N}{N_h}\sigma_2(\lambda) = \frac{1}{2\pi}\left[\phi'(\lambda) + \frac{2N}{N_p}\int_{[C_1]}[\phi'(2(\lambda-\mu)) + \tfrac{1}{3}\phi'(\tfrac{2}{3}(\lambda-\mu))]\sigma_1(\mu)\,\mathrm{d}\mu\right.$$
$$\left. - \frac{N}{N_p}\int_{[C_1]}[2\phi'(\lambda-\mu) + \tfrac{1}{2}\phi'(\tfrac{1}{2}(\lambda-\mu))]\sigma_2(\mu)\,\mathrm{d}\mu\right] \qquad (4.24b)$$

and the integration regions can be determined by n_e and s^z:

$$n_e = \frac{N_p}{N} + 2\int_{[C_1]} \sigma_1(\mu)\,\mathrm{d}\mu \qquad (4.25a)$$

$$s^z = \frac{n_e}{2} - 2\int_{[C_2]} \sigma_2(\mu)\,\mathrm{d}\mu. \qquad (4.25b)$$

These equations can be solved numerically for any filling n_e and magnetization s^z.

If we take a system where the integration limits turn out to be,

$$[C_1] = [C_2] = (-\infty, \infty)$$

our equations can be solved analytically using the Fourier transform and some identities in appendix B.

Let

$$\sigma_j(\lambda) = \int_{-\infty}^{\infty} \hat{\sigma}_j(\alpha)\mathrm{e}^{\mathrm{i}\alpha\lambda}\,\mathrm{d}\alpha \qquad (4.26)$$

then we obtain,

$$\hat{\sigma}_1(\alpha) = \frac{N_h}{N}\frac{\mathrm{e}^{-|\alpha|/2}}{2\cosh\frac{\alpha}{2}} - \frac{N_p}{N}\frac{\mathrm{e}^{-|\alpha|/2}}{4\cosh^2\frac{\alpha}{2}} \qquad (4.27a)$$

$$\hat{\sigma}_2(\alpha) = \frac{N_h}{N}\frac{\mathrm{e}^{-|\alpha|/2}}{4\cosh^2\frac{\alpha}{2}} + \frac{N_p}{N}\frac{\mathrm{e}^{|\alpha|/2}}{8\cosh^3\frac{\alpha}{2}}. \qquad (4.27b)$$

These expressions can be used to determinate the main parameters of our system

$$\frac{r}{N} = 2\int \sigma_1(\alpha)\,\mathrm{d}\alpha = 2\hat{\sigma}_1(0) = \frac{2N_h - N_p}{2N} \qquad (4.28a)$$

$$\frac{s}{N} = 2\int \sigma_2(\alpha)\,\mathrm{d}\alpha = 2\hat{\sigma}_2(0) = \frac{2N_h + N_p}{4N} \qquad (4.28b)$$

and using (4.25a), (4.25b) we find,

$$n_e = \frac{2N_h + N_p}{2N} \tag{4.29a}$$

$$s^z = 0. \tag{4.29b}$$

We must note that for $N_p = 0$, we have $n_e = 1$ and $s^z = 0$, this constitutes a antiferromagnetic state with half-filling.

From (4.20), the energy is given by

$$\frac{E}{N} = J\left[\frac{N_h}{N}(1 - 2\ln 2) + \frac{N_p}{N}\left(\frac{3\pi}{2} - 3\right)\right] \tag{4.29c}$$

and, for $N_p = 0$ we obtain the result of the t–J model.

As we know, it must verify

$$\frac{N_p}{N} \leqslant n_e \leqslant \frac{N_h + 2N_p}{N} \tag{4.29d}$$

then, from (4.29a), this solution will only be true if,

$$\frac{N_p}{N_h} \leqslant 2. \tag{4.29e}$$

5. The excitation spectrum

We start again from the BAE (4.6a), (4.6b). The more general string solutions will be of the form,

$$v^{(1)}_{k,(m)} = v^{(1)}_{k,M} + \mathrm{i}m \qquad m = -M, \ldots, M \tag{5.1a}$$

$$v^{(2)}_{k,(m)} = v^{(2)}_{k,M'} + \mathrm{i}m \qquad m = -M', \ldots, M' \tag{5.1b}$$

with M and M' integer or half-odd integer.

Following the same method as before, we multiply the equations of the same string, and we obtain that their centre $v^{(i)}_{k,M'}$, $i = 1, 2$, verifies the equations

$$2N_h \arctan\frac{v^{(1)}_{k,M}}{M + \frac{1}{2}} = 2\pi I^{(1)}_{k,M} + \sum_{M''}\sum_{j=1}^{\nu^{(2)}_{M''}} \Phi_{M,M''}(v^{(1)}_{k,M} - v^{(2)}_{j,M''}) \tag{5.2a}$$

$$\begin{aligned}-2N_p \arctan\frac{v^{(2)}_{k,M}}{M + \frac{1}{2}} &= -2\pi I^{(2)}_{k,M} - \sum_{M'}\sum_{j=1}^{\nu^{(2)}_{M'}} \Psi_{M,M'}(v^{(2)}_{k,M} - v^{(2)}_{j,M'}) \\ &\quad + \sum_{M''}\sum_{l=1}^{\nu^{(1)}_{M''}} \Phi_{M,M''}(v^{(2)}_{k,M} - v^{(1)}_{l,M''})\end{aligned} \tag{5.2b}$$

where $\nu^{(i)}_M$ is the number of strings in the i level, and the Φ and Ψ functions are defined in appendix B. A solution of (5.2a), (5.2b), is determined by specifying sets of integers or half-odd integers $\{I^{(1)}_{k,M}\}$ and the regions where the $\{C\}$ roots are distributed.

As we did before, we define the functions,

$$F^{(1)}_M(\lambda) = \frac{N_h}{\pi}\arctan\frac{\lambda}{M + \frac{1}{2}} - \frac{1}{2\pi}\sum_{M''}\sum_{j=1}^{\nu^{(2)}_{M''}} \Phi_{M,M''}(\lambda - v^{(2)}_{j,M''}) \tag{5.3a}$$

$$F^{(2)}_M(\lambda) = \frac{N_p}{\pi}\arctan\frac{\lambda}{M + \frac{1}{2}} - \frac{1}{2\pi}\sum_{M'}\sum_{j=1}^{\nu^{(2)}_{M'}} \Psi_{M,M'}(\lambda - v^{(2)}_{j,M'})$$

$$+\frac{1}{2\pi}\sum_{M''}\sum_{j=1}^{\nu^{(1)}_{M''}}\Phi_{M,M''}(\lambda - v^{(1)}_{j,M''}) \tag{5.3b}$$

that are monotonically increasing and reach integer or half-odd integers values when λ takes the value of a root.

By counting the number of roots, and calling $H^{(i)}_M$ the number or holes in the sea of M-strings at level i, we have,

$$2I^{(i)}_{max,M} + 1 = \nu^{(i)}_M + H^{(i)}_M \tag{5.4}$$

and supposing that the centres of the strings are distributed along the real numbers,

$$2I^{(i)}_{max,M} + 1 = F^{(i)}_M(\infty) \tag{5.5}$$

and then, using (5.3)–(5.5), we obtain,

$$\nu^{(1)}_M + H^{(1)}_M = N_h - 2\sum_{M''\geqslant 0} K(M, M'')\nu^{(2)}_{M''} \tag{5.6a}$$

$$\nu^{(2)}_M + H^{(2)}_M = N_p - 2\sum_{M'\geqslant 0} J(M, M')\nu^{(2)}_{M'} + 2\sum_{M''\geqslant 0} K(M, M'')\nu^{(1)}_{M''} \tag{5.6b}$$

where

$$J(M_1, M_2) = \begin{cases} 2M_1 + \frac{1}{2} & \text{if} \quad M_1 = M_2 \\ 2\min(M_1, M_2) + 1 & \text{if} \quad M_1 \neq M_2 \end{cases} \tag{5.7}$$

and

$$K(M_1, M_2) = \begin{cases} M_2 + \frac{1}{2} & \text{if} \quad M_2 + \frac{1}{2} \leqslant M_1 \\ M_1 + \frac{1}{2} & \text{if} \quad M_2 + \frac{1}{2} > M_1. \end{cases} \tag{5.8}$$

Besides, the number of roots, obviously, must verify,

$$r = \sum_{M\geqslant 0}(2M + 1)\nu^{(1)}_M \tag{5.9a}$$

$$s = \sum_{M\geqslant 0}(2M + 1)\nu^{(2)}_M \tag{5.9b}$$

and now, we can apply (5.6*a*), (5.6*b*) and (5.9*a*), (5.9*b*) to the ground state, according to which type of string we considered that forms it.

If we suppose, as before, that the ground state is formed with two seas of two-strings, we have,

$$H^{(1)}_{\frac{1}{2}} = N_h - \nu^{(1)}_{\frac{1}{2}} - \nu^{(2)}_0 - 2\sum_{M''\geqslant\frac{1}{2}} \nu^{(2)}_{M''} \tag{5.10a}$$

$$H^{(2)}_{\frac{1}{2}} = N_p - 2\nu^{(2)}_0 - 4\sum_{M'\geqslant\frac{1}{2}} \nu^{(2)}_{M'} - \nu^{(1)}_0 - 2\sum_{M''\geqslant\frac{1}{2}} \nu^{(1)}_{M''} \tag{5.10b}$$

and then, with these results, we obtain from (4.28*a*), (4.28*b*),

$$H^{(1)}_{\frac{1}{2}} = H^{(2)}_{\frac{1}{2}} = 0 \tag{5.11}$$

that is to say, we have two two-string seas without holes.

Under these hypothesis we can analyse different types of excitations by introducing holes in the ground state and keeping constant some observables. The one that we are going to consider maintains constant the electron number and the magnetization

$$n_e = \text{const.} \qquad s_z = \text{const.} \tag{5.12}$$

then, by imposing (5.9*a*), (5.9*b*) and (5.10*a*), (5.10*b*), we obtain that the state must be characterized having two real roots and a hole in level (1)

$$\nu_0^{(1)} = 2 \qquad H_{\frac{1}{2}}^{(1)} = 1 \tag{5.13}$$

and besides,

$$\begin{aligned} &\nu_0^{(2)} = 0 \qquad \nu_{\frac{1}{2}}^{(1)} = \frac{2N_h - N_p}{4} - 1 \qquad \nu_{M\geqslant\frac{1}{2}}^{(1)} = 0 \\ &H_{\frac{1}{2}}^{(2)} = 0 \qquad \nu_{\frac{1}{2}}^{(2)} = \frac{2N_h + N_p}{8} \qquad \nu_{M\geqslant\frac{1}{2}}^{(2)} = 0. \end{aligned} \tag{5.14}$$

These conditions correspond to one of the two states, the first one is obtained from the ground state by changing two consecutive sites with spin up to a new state with the two sites in spin down, the second one is a state with one of the two sides without electrons and the other with a pair.

Under our hypothesis about the ground state, we can calculate the contribution of every hole and real root in both levels, to the energy of an excited state compared with the energy of the ground state. Using (4.17), the equations (4.24*a*), (4.24*b*) must change to,

$$\begin{aligned} \frac{N}{N_h}\sigma_1(\lambda) + \frac{1}{N_h}\sum_{h_1}\delta(\lambda - \theta_{h_1}) = \frac{1}{2\pi}\bigg[&\phi'(\lambda) - \frac{1}{N_h}\sum_{r_2}\phi'(\lambda - \theta_{r_2}) \\ &- \frac{2N}{N_h}\int_{[C_2]}[\phi'(2(\lambda-\mu)) + \tfrac{1}{3}\phi'(\tfrac{2}{3}(\lambda-\mu))]\sigma_2(\mu)\,\mathrm{d}\mu\bigg] \end{aligned} \tag{5.15a}$$

$$\begin{aligned} \frac{N}{N_p}\sigma_2(\lambda) + \frac{1}{N_p}\sum_{h_2}\delta(\lambda - \theta_{h_2}) = \frac{1}{2\pi}\bigg[&\phi'(\lambda) + \frac{1}{N_p}\sum_{r_1}\phi'(\lambda - \theta_{r_1}) \\ &- \frac{2}{N_p}\sum_{r_2}[\phi'(2(\lambda-\theta_{r_2})) + \tfrac{1}{3}\phi'(\tfrac{2}{3}(\lambda-\theta_{r_2}))] \\ &+ \frac{2N}{N_p}\int_{[C_1]}[\phi'(2(\lambda-\mu)) + \tfrac{1}{3}\phi'(\tfrac{2}{3}(\lambda-\mu))]\sigma_1(\mu)\,\mathrm{d}\mu \\ &- \frac{N}{N_p}\int_{[C_1]}[2\phi'(\lambda-\mu) + \tfrac{1}{2}\phi'(\tfrac{1}{2}(\lambda-\mu))]\sigma_2(\mu)\,\mathrm{d}\mu\bigg] \end{aligned} \tag{5.15b}$$

where θ_{h_i} parametrizes the holes at level i and θ_{r_i} the real roots.

The system can be solved as before, by using the Fourier transform, and we obtain that the distribution functions can be written as,

$$\hat{\sigma}_1(\alpha) = \hat{\sigma}_{1(0)}(\alpha) + \hat{\sigma}_{1(n)}(\alpha) \tag{5.16a}$$

$$\hat{\sigma}_2(\alpha) = \hat{\sigma}_{2(0)}(\alpha) + \hat{\sigma}_{2(n)}(\alpha) \tag{5.16b}$$

with

$$\hat{\sigma}_{1(0)}(\alpha) = \frac{N_h}{N}\frac{\mathrm{e}^{-|\alpha|}}{1+\mathrm{e}^{-|\alpha|}} - \frac{N_p}{N}\frac{\mathrm{e}^{-3|\alpha|/2}}{(1+\mathrm{e}^{-|\alpha|})^2} \tag{5.17a}$$

$$\hat{\sigma}_{2(0)}(\alpha) = \frac{N_h}{N}\frac{\mathrm{e}^{-3|\alpha|/2}}{(1+\mathrm{e}^{-|\alpha|})^2} + \frac{N_p}{N}\frac{\mathrm{e}^{-|\alpha|}}{(1+\mathrm{e}^{-|\alpha|})^3} \tag{5.17b}$$

$$\hat{\sigma}_{1(n)}(\alpha) = \frac{1}{N}\bigg[\sum_{h_2}\frac{\mathrm{e}^{-|\alpha|/2}}{(1+\mathrm{e}^{-|\alpha|})^2}\mathrm{e}^{-\mathrm{i}\alpha\theta_{h_2}} - \sum_{h_1}\frac{1}{(1+\mathrm{e}^{-|\alpha|})}\mathrm{e}^{-\mathrm{i}\alpha\theta_{h_1}} - \sum_{r_1}\frac{\mathrm{e}^{-3|\alpha|/2}}{(1+\mathrm{e}^{-|\alpha|})^2}\mathrm{e}^{-\mathrm{i}\alpha\theta_{r_1}}\bigg] \tag{5.17c}$$

$$\hat{\sigma}_{2(n)}(\alpha) = \frac{1}{N}\bigg[\sum_{h_1}\frac{\mathrm{e}^{-|\alpha|/2}}{(1+\mathrm{e}^{-|\alpha|})^2}\mathrm{e}^{-\mathrm{i}\alpha\theta_{h_1}} - \sum_{h_2}\frac{1}{(1+\mathrm{e}^{-|\alpha|})^3}\mathrm{e}^{-\mathrm{i}\alpha\theta_{h_2}}$$

$$+\sum_{r_1} \frac{e^{-|\alpha|}}{(1+e^{-|\alpha|})^3} e^{-i\alpha\theta_{r_1}} + \sum_{r_2} \frac{e^{-|\alpha|/2}}{(1+e^{-|\alpha|})} e^{-i\alpha\theta_{r_2}} \Bigg]. \tag{5.17d}$$

The contribution to the energy per site is,

$$\Delta e \equiv \frac{\Delta E}{N} = \int \phi'(\mu)\sigma_{1(n)}(\mu)\,d\mu + \frac{1}{N}\sum_{r_1} \frac{1}{(\theta_{r_1}^2 + \frac{1}{4})}. \tag{5.18}$$

From this expression, we observe that the holes in both levels and only the real roots of the first level contribute to the energy. A straightforward calculation shows the following rules:

(i) every hole in the first level gives a contribution $\Delta e = -\varepsilon_1(\theta_{h_1})$;
(ii) every hole in the second level gives a contribution $\Delta e = \varepsilon_2(\theta_{h_2})$;
(iii) every real root in the first level gives a contribution

$$\Delta e = \frac{1}{N}\,\mathrm{sech}(\pi\theta_{r_1}) + \varepsilon_2(\theta_{r_1})$$

(iv) every real root in the second level gives a contribution $\Delta e = 0$, being

$$\varepsilon_n(v) = \frac{1}{nN}\int_0^\infty \frac{\cos\alpha v}{e^{\frac{\alpha}{2}}\cosh^n\frac{\alpha}{2}}\,d\alpha. \tag{5.19}$$

We can apply these rules to the first example, where we have held constant the number of electrons and magnetization with respect to the ground state and it is characterized by two real roots and a hole. We can suppose that the roots and the hole are parametrized by the same value θ, it is said, the state is coming by changing a two-string of the ground state by two real roots, which leaves a hole in the two-string sea. Performing the integrals in Δe, we obtain for the state,

$$\begin{aligned}\Delta e = \frac{1}{N}\Bigg[&\frac{2}{\cosh(\pi\theta)} + 2\mathrm{Re}\left[\Psi\left(\frac{1}{2}+i\theta\right)\right] - 2\mathrm{Re}\left[\Psi\left(\frac{1}{4}+i\frac{\theta}{2}\right)\right]\\ &-4\theta\left(\mathrm{Im}\left[\Psi\left(\frac{1}{2}+i\theta\right)\right] - \mathrm{Im}\left[\Psi\left(\frac{1}{4}+i\frac{\theta}{2}\right)\right]\right)\\ &-2\mathrm{Re}\left[\Psi\left(i\frac{\theta}{2}\right)\right] + 2\mathrm{Re}\left[\Psi(i\theta)\right] - 2(1+2\ln 2)\Bigg]\end{aligned} \tag{5.20}$$

where $\Psi(x)$ is the derivative of the logarithm of the Euler gamma function.

Taking an alternating chain $N_h = N_p = N/2$, we can represent the energy given by (5.20). The results are shown in figure 1. As we can see, Δe is null out of an interval around the origin. In conclusion, there is not an energy gap between the excited and the ground state.

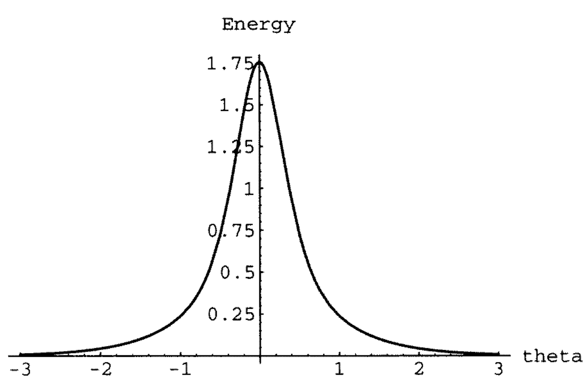

Figure 1. Energy of the excited state.

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Appendix A

Using the relation of commutation (3.6c) we have

$$C_{a_1}(\lambda_1)C_{a_2}(\lambda_2)\ldots C_{a_r}(\lambda_r) = C_{b_2}(\lambda_2)C_{b_3}(\lambda_3)\ldots C_{b_r}(\lambda_r)C_{b_1}(\lambda_1)G(\lambda_1,\vec{\lambda})^{b_1,\ldots,b_r}_{a_1,\ldots,a_r} \tag{A.1}$$

with

$$G(u,\vec{\lambda})^{b_1,\ldots,b_r}_{a_1,\ldots,a_r} = r(u-\lambda_1)^{i_1,a_1}_{b_1,a}r(u-\lambda_2)^{i_2,a_2}_{b_2,i_1}\ldots r(u-\lambda_r)^{a,a_r}_{b_r,i_{r-1}}. \tag{A.2}$$

Taking

$$M^{(j)}(\lambda_j) = G(\lambda_j,\vec{\lambda})\cdot G(\lambda_{j-1},\vec{\lambda})\cdot\ldots\cdot G(\lambda_1,\vec{\lambda}) \tag{A.3}$$

we have the relation

$$C_{a_1}(\lambda_1)C_{a_2}(\lambda_2)\ldots C_{a_r}(\lambda_r) = C_{b_k}(\lambda_k)\ldots C_{b_r}(\lambda_r)C_{b_1}(\lambda_1)\ldots C_{b_{k-1}}(\lambda_{k-1})M^{(k-1)}(\lambda_{k-1})^{b_1,\ldots,b_r}_{a_1,\ldots,a_r}. \tag{A.4}$$

Appendix B

In this appendix, we are going to give brief remarks about the function,

$$\phi(x) \equiv 2\arctan(x) = -\mathrm{i}\ln\frac{1+\mathrm{i}x}{1-\mathrm{i}x} \tag{B.1}$$

that takes values in the interval $-\pi$ to $+\pi$ when x goes from $-\infty$ to ∞,

The function has the following properties, that can be proved by a straightforward calculation:

$$\phi(x+\mathrm{i}) + \phi(x-\mathrm{i}) = \pi + \phi\left(\frac{x}{2}\right) \tag{B.2}$$

$$\phi(x+2\mathrm{i}) + \phi(x-2\mathrm{i}) = \phi\left(\frac{x}{3}\right) - \phi(x) \tag{B.3}$$

$$\phi\left(x+\frac{\mathrm{i}}{2}\right) + \phi\left(x-\frac{\mathrm{i}}{2}\right) = \phi\left(\frac{2x}{3}\right) + \phi(2x). \tag{B.4}$$

The Fourier transform that we have used

$$\rho(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\rho}(\alpha)\mathrm{e}^{\mathrm{i}\alpha\lambda}\,\mathrm{d}\alpha \qquad \hat{\rho}(\alpha) = \int_{-\infty}^{\infty}\rho(\lambda)\mathrm{e}^{-\mathrm{i}\alpha\lambda}\,\mathrm{d}\lambda. \tag{B.5}$$

The derivative of function ϕ is

$$\frac{\mathrm{d}\phi(\lambda)}{\mathrm{d}x} = \frac{2}{1+\lambda^2} \tag{B.6}$$

and then, the Fourier transform,

$$\int_{-\infty}^{\infty}\phi'(\lambda)\mathrm{e}^{-\mathrm{i}\alpha\lambda}\,\mathrm{d}\lambda = \mathrm{e}^{-|\alpha|}. \tag{B.7}$$

It is convenient to define the functions

$$\Psi_{M_1,M_2}(x) = 2\sum_{n=|M_2-M_1|}^{M_2+M_1}\left[\arctan\frac{x}{n} + \arctan\frac{x}{n+1}\right] \tag{B.8}$$

and

$$\Phi_{M_1,M_2}(x) = 2\sum_{n=-M_1}^{M_1}\arctan\frac{x+\mathrm{i}n}{M_2+\frac{1}{2}}. \tag{B.9}$$

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